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ON GRADED WEAKLY CLASSICAL PRIMARY SUBMODULES

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Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring and M a graded R-module. In this paper, we introduce the concept of graded weakly classical primary submodules. Various properties of graded weakly classical primary submodules are considered.

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1. Introduction and Preliminaries

Graded prime ideals and graded primary ideals of a commutative graded ring have been introduced and studied by Refai and Al-Zoubi in [13]. Graded weakly prime ideals and graded weakly primary ideals of a commutative graded ring have been introduced and studied in [7, 8]. Graded prime submodules and graded primary submodules of graded modules over graded commutative rings have been studied by various authors; (see, for example [5, 9, 12]. Graded weakly prime submodules and graded weakly primary submodules of graded modules over graded commutative rings have been introduced and studied in [1, 6]. Graded classical prime submodules and graded classical primary submodules of graded modules over graded commutative rings have been introduced and studied in [2, 3]. Graded weakly classical prime submodules of graded modules over graded commutative rings have been introduced and studied in [4]. Here we introduce the concept of graded weakly classical primary submodules. A number of results concerning of graded weakly classical primary submodules are given (see sec. 2).

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [10] and [11] for these basic properties and more information on graded rings and modules. Let G be a group with identity e and R be a commutative ring with identity 1_R. Then R is a G-*graded ring* if there exist additive subgroups R_g of R such that $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called to be *homogeneous* of degree g where the R_g's are additive subgroups of R indexed by the elements $g \in G$. If $x \in R$, then x can be written uniquely as $\sum_{g\in G} x_g$, where x_g is the component of x in R_g. Also we write, $h(R) = \bigcup_{g\in G} R_g$. Moreover, R_e is a subring of R and $1_R \in R_e$. Let I be an ideal of R. Then I is called a *graded ideal* of (R, G) if $I = \bigoplus_{g\in G}(I \cap R_g)$. Thus, if $x \in I$, then $x = \sum_{g\in G} x_g$ with $x_g \in I$. An ideal of a G-graded ring need not be G-graded.

Let R be a G-graded ring and M an R-module. We say that M is a *G-graded R-module* (or *graded R-module*) if there exists a family of subgroups $\{M_g\}_{g\in G}$ of M such that $M = \underset{g\in G}{\oplus} M_g$ (as abelian groups) and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Here, $R_g M_h$ denotes the additive subgroup of M consisting of all finite sums of elements $r_g s_h$ with $r_g \in R_g$ and $s_h \in M_h$. Also, we write $h(M) = \underset{g\in G}{\cup} M_g$ and the elements of $h(M)$ are called to be *homogeneous*. Let $M = \underset{g\in G}{\oplus} M_g$ be a graded R-module and N a submodule of M. Then N is called a *graded submodule* of M if $N = \underset{g\in G}{\oplus} N_g$ where $N_g = N \cap M_g$ for $g \in G$. In this case, N_g is called the *g-component* of N.

Let R be a G-graded ring, M be a graded R-module and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fraction $S^{-1}R$ is a graded ring which is called graded ring of fractions. Indeed, $S^{-1}R = \underset{g\in G}{\oplus}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S \ \textit{and}\ g = (\deg s)^{-1}(\deg r)\}$. The module of fraction $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called module of fractions, if $S^{-1}M = \underset{g\in G}{\oplus}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S \ \textit{and}\ g = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \underset{g\in G}{\cup}(S^{-1}R)_g$ and $h(S^{-1}M) = \underset{g\in G}{\cup}(S^{-1}M)_g$.

Let R be a G-graded ring and M a graded R-module. The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $x \in R$ such that for each $g \in G$ there exists $n_g > 0$ with $x_g^{n_g} \in I$. Note that, if r is a homogeneous element, then $r \in Gr(I)$ if and only if $r^n \in I$ for some $n \in \mathbb{N}$(see [13].) A proper graded ideal P of R is said to be *graded prime* (resp. *graded weakly prime*) *ideal* if whenever $r, s \in h(R)$ with $rs \in P$ (resp. $0 \neq rs \in P$), then either $r \in P$ or $s \in P$(see [7, 13].)

A proper graded ideal P of R is said to be *graded primary* (resp. *graded weakly primary*) *ideal* if whenever $r, s \in h(R)$ with $rs \in P$ (resp. $0 \neq rs \in P$), then either $r \in P$ or $s \in Gr(P)$, (see [8, 13].)

A proper graded submodule N of M is said to be *graded prime* (resp. *graded weakly prime*) *submodule* if whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$ (resp. $0 \neq rm \in N$), then either $r \in (N :_R M)$ or $m \in N$(see [5, 6].)

The *graded radical* of a graded submodule N of a graded R-module M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$ (see [9].)

A proper graded submodule N of M is called a *graded classical prime*(resp. *graded weakly classical prime*) submodule if whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$ (resp. $0 \neq rsm \in N$), then either $rm \in N$ or $sm \in N$ (see [2, 4].)

Let I be a graded ideal of R and $h \in G$. The set $\{r \in R_h : r^n \in I$ for some positive integer $n\}$ is a subgroup of R_h and is called h-radical of I, denoted by $hGr(I)$. Clearly, $I_h \subseteq hGr(I)$ and if $r \in R_h$ with $r \in Gr(I)$, then $r \in hGr(I)$. Let N be a graded submodule of M and let $g \in G$. We say that

N_g is a g-primary (resp. *weakly g-primary*) submodule of the R_e-module M_g if $N_g \neq M_g$; and whenever $r \in R_e$ and $m \in M_g$ with $rm \in N_g$ (resp. $0 \neq rm \in N_g$), then either $m \in N_g$ or $r^n \in (N_g :_{R_e} M_g)$ for some positive integer n (that is $r \in eGr((N_g :_{R_e} M_g))$. We say that N is a graded primary (resp. *graded weakly primary)submodule* of M if $N \neq M$; and whenever $r \in h(R)$ and $m \in h(M)$ with $rm \in N$ (resp. $0 \neq rm \in N$), then either $m \in N$ or $r \in Gr((N :_R M))$ (see [1, 12].) A graded R-module M is called *graded cyclic* if $M = Rm$ where $m \in h(M)$ (see [10].)

2. RESULTS

Definition 2.1. [3, Definition 1] Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $g \in G$.

(i) We say that N_g is a classical g-primary submodule of the R_e-module M_g if $N_g \neq M_g$; and whenever $r, s \in R_e$ and $m \in M_g$ with $rsm \in N_g$, then either $rm \in N_g$ or $s^n m \in N_g$ for some positive integer n (that is $s \in eGr((N_g :_{R_e} m))$.

(ii) We say that N is a graded classical primary submodule of M if $N \neq M$; and whenever $r, s \in h(R)$ and $m \in h(M)$ with $rsm \in N$, then either $rm \in N$ or $s^n m \in N$ for some positive integer n.

Definition 2.2. Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $g \in G$.

(i) We say that N_g is a weakly classical g-primary submodule of the R_e-module M_g if $N_g \neq M_g$; and whenever $r, s \in R_e$ and $m \in M_g$ with $0 \neq rsm \in N_g$, then either $rm \in N_g$ or $s^n m \in N_g$ for some positive integer n (that is $s \in eGr((N_g :_{R_e} m))$.

(ii) We say that N is a graded weakly classical primary submodule of M if $N \neq M$; and whenever $r, s \in h(R)$ and $m \in h(M)$ with $0 \neq rsm \in N$, then either $rm \in N$ or $s^n m \in N$ for some positive integer n.

Clearly, a graded classical primary submodule of M (resp. a classical g-primary R_e-submodule of M_g) is a graded weakly classical primary submodule of M (resp. weakly classical g-primary R_e-submodule of M_g). However, since $\{0\}$ is always a graded weakly classical primary submodule of M (resp. a weakly g-classical primary R_e-submodule of M_g) (by definition), a graded weakly classical primary submodule of M (resp. a weakly classical g-primary R_e-submodule of M_g) need not be graded classical primary (resp. classical g-primary).

By definition, every graded weakly classical prime submodule of a graded module is graded weakly classical primary. However, the converse is not true in general. The example of this is given below.

Example 2.3. Let $G = (\mathbb{Z}, +)$ and $R = (\mathbb{Z}, +, .)$. Define

$$R_g = \left\{ \begin{array}{cc} \mathbb{Z} & \text{if } g = 0 \\ 0 & \text{otherwise} \end{array} \right\}$$

Then R is a G-graded ring. Let $M = \mathbb{Z} \times \mathbb{Z}$. Then M is a G-graded R-module with

$$M_g = \left\{ \begin{array}{cc} \mathbb{Z} \times \{0\} & \text{if } g = 0 \\ \{0\} \times \mathbb{Z} & \text{if } g = 1 \\ \{0\} \times \{0\} & \text{otherwise} \end{array} \right\}.$$

Let $N = 9\mathbb{Z} \times \{0\}$. Then N is a graded submodule of M. For $3 \in R_0$ and $(2,0) \in M_0$, $0 \neq 3^2(2,0) \in N$, but $3(2,0) \notin N$. Hence N is not graded weakly classical prime. But, N is graded weakly classical primary.

Theorem 2.4. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If N is a graded weakly primary submodule of M, then N is a graded weakly classical primary submodule of M.*

Proof. Assume that N is a graded weakly primary submodule of M. Let $r, s \in h(R)$ and $m_g \in h(M)$ such that $0 \neq rsm_g \in N$ and $sm_g \notin N$. Since N is a graded weakly primary submodule of M and $sm_g \notin N$, we have $r \in Gr((N :_R M))$ and hence $r^n M \in N$ for some positive integer n. So $r^n m_g \in N$. Therefore N is a graded weakly classical primary submodule of M. □

However, the converse of Theorem 2.4 is not true in general, see Example 2.3.

Theorem 2.5. *Let R be a G-graded ring, M a graded cyclic R-module and N a graded submodule of M. Then N is a graded weakly primary submodule of M if and only if N is a graded weakly classical primary submodule of M.*

Proof. ($\Rightarrow$)By Theorem 2.4.

($\Leftarrow$) Let $M = Rx$ for some $x \in h(M)$ and N be a graded weakly classical primary submodule of M. Assume that $r \in h(R)$ and $m \in h(M)$ such that $0 \neq rm \in N$ and $m \notin N$. Then there exists $t \in h(R)$ such that $m = tx$. Since N is a graded weakly classical primary submodule of M, $0 \neq rtx \in N$ and $tx \notin N$, we conclude that $r^n x \in N$ for some positive integer n. Hence $r^n M \subseteq N$ and so $r \in Gr((N :_R M))$. Therefore, N is graded weakly primary submodule of M. □

Theorem 2.6. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If N is a graded weakly classical primary submodule of M, then N_g is a weakly classical g-primary R_e-submodule of M_g for every $g \in G$.*

Proof. Suppose that N is a graded weakly classical primary submodule of M. For $g \in G$ assume that $0 \neq rsm \in N_g \subseteq N$ where $r, s \in R_e$ and $m \in M_g$. Since N is a graded weakly classical primary submodule of M, we have either $rm \in N$ or $s^n m \in N$ for some positive integer n and hence either $rm \in N_g$ or $s^n m \in N_g$. Thus N_g is a weakly classical g-primary R_e-submodule of M_g. □

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If $(N :_R m_g)$ is a graded weakly primary ideal of R for every $m \in h(M)$, then N is a graded weakly classical primary submodule of M.*

Proof. Let $r, s \in h(R)$ and $m_g \in h(M)$ such that $0 \neq rsm_g \in N$ and $m_g \notin N$. Then $0 \neq rs \in (N :_R m_g)$ and since $(N : m_g)$ is a graded weakly primary ideal, either $r \in (N :_R m_g)$ or $s^n \in (N :_R m_g)$ for some positive integer n and then either $rm \in N$ or $s^n m \in N$ and hence N is a graded weakly classical primary R-submodule of M. □

Theorem 2.8. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M . If N is a graded weakly classical primary R-submodule of M and $m_g \in h(M) - N$ such that $Ann_R(m_g) = 0$, then $(N :_R m_g)$ is a graded weakly primary ideal of R.*

Proof. By [5 , Lemma 2.1], $(N :_R m_g)$ is a graded ideal of R. Let $r, s \in h(R)$ such that $0 \neq rs \in (N :_R m)$. Then $0 \neq rsm_g \in N$. Since N is a graded weakly classical primary, either $rm \in N$ or $s^n m \in N$ for some positive integer n and then either $r \in (N :_R m_g)$ or $s^n \in (N :_R m_g)$. Hence, $(N :_R m_g)$ is a graded weakly primary ideal of R. □

Let M and M' be two graded R-modules. A homomorphism of graded R-modules $\varphi : M \to M'$ is a homomorphism of R-modules verifying $\varphi(M_g) \subseteq M'_g$ for every $g \in G$,.

Theorem 2.9. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be an epimorphism of graded modules. Let N be a graded submodule of M such that $Ker\varphi \subseteq N$. If N is a graded weakly classical primary submodule of M, then $\varphi(N)$ is a graded weakly classical primary submodule of M'.*

Proof. Suppose that N is a graded weakly classical primary submodule of M and let $r, s \in h(R)$ and $m' \in h(M')$ such that $0 \neq rsm' \in \varphi(N)$ and $rm' \notin \varphi(N)$. Since $rsm' \in \varphi(N)$, there exists $t \in N \cap h(M)$ such that $\varphi(t) = rsm'$. Since $m' \in h(M')$ and φ is an epimorphism, there exists $m \in h(M)$ such that $\varphi(m) = m'$. Thus $\varphi(t) = rs\varphi(m)$ and hence $\varphi(t - rsm) = 0$. So $t - rsm \in Ker\varphi \subseteq N$ and hence $0 \neq rsm \in N$. Since N is a graded weakly classical primary submodule of M and $rm \notin N$, $s^n m \in N$ for some $n \in \mathbb{Z}^+$. So $r^n m' \in \varphi(N)$. Thus $\varphi(N)$ is a graded weakly classical primary submodule of M' □

Theorem 2.10. *Let R be a G-graded ring and M, M' be two graded R-modules and $\varphi : M \to M'$ be an isomorphism of graded modules. Let N' be a graded submodule of M'. Then N' is a graded weakly classical primary submodule of M' if and only if $\varphi^{-1}(N')$ is a graded weakly classical primary submodule of M.*

Proof. ($\Rightarrow$) Suppose that N' is a graded weakly classical primary submodule of M' and let $r, s \in h(R)$ and $m \in h(M)$ such that $0 \neq rsm \in \varphi^{-1}(N')$ and $rm \notin \varphi^{-1}(N')$. Since φ is a monomorphism, $0 \neq \varphi(rsm) = rs\varphi(m) \in N'$. Since N' is a graded weakly classical primary submodule of M' and $r\varphi(m) = \varphi(rm) \notin N'$, we have $s^n\varphi(m) = \varphi(s^n m) \in N'$ for some $n \in \mathbb{Z}^+$ and hence $s^n m \in \varphi^{-1}(N')$. Thus $\varphi^{-1}(N')$ is a graded weakly classical primary submodule of M.

($\Leftarrow$)Suppose that $\varphi^{-1}(N')$ is a graded weakly classical primary submodule of M and let r,$s \in h(R)$ and $m' \in h(M')$ such that $0 \neq rsm' \in N'$ and $rm' \notin N'$. Since φ is an epimorphism, there exists $m \in h(M)$ such that $\varphi(m) = m'$. Thus $0 \neq rs\varphi(m) = \varphi(rsm) \in N'$. So $0 \neq rsm \in \varphi^{-1}(N')$. Since $\varphi^{-1}(N')$

is a graded weakly classical primary submodule of M and $rm \notin \varphi^{-1}(N')$, we have $s^n m \in \varphi^{-1}(N')$ for some $n \in \mathbb{Z}^+$, and so $\varphi(s^n m) = s^n \varphi(m) \in N'$. Therefore N' is a graded weakly classical primary submodule. □

A graded zero-divisor on a graded R-module M is an element $r \in h(R)$ for which there exists $m \in h(M)$ such that $m \neq 0$ but $rm = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}Zdv_R(M)$.

The following result studies the behavior of graded weakly classical primary submodules under localization.

Theorem 2.11. *Let R be a G-graded ring, M a graded R-module and $S \subseteq h(R)$ a multiplication closed subset of R. Then the following hold:*

(i) *If N is a graded weakly classical primary R-submodule of M and $S \cap (N : M) = \phi$, then $S^{-1}N$ is a graded weakly classical primary submodule of $S^{-1}M$.*

(ii) *If $S^{-1}N$ is a graded weakly classical primary submodule of $S^{-1}M$ such that $G\text{-}Zdv_R(N) \bigcap S = \phi$ and $G\text{-}Zdv_R(M/N) \bigcap S = \phi$, then N is a graded weakly classical primary R-submodule of M.*

Proof. (i) Suppose that N be a graded weakly classical primary R-submodule of M and $S \cap (N : M) = \phi$. Let $0 \neq \frac{p_1}{r}\frac{p_2}{s}\frac{m}{t} \in S^{-1}N$ for some $\frac{p_1}{r}, \frac{p_2}{s} \in h(S^{-1}R)$ and for some $\frac{m}{t} \in h(S^{-1}M)$. Then there exists $u \in S$ such that $up_1p_2m \in N$. If $up_1p_2m = 0$, then $\frac{p_1}{r}\frac{p_2}{s}\frac{m}{t} = \frac{up_1p_2m}{urst} = \frac{0}{1}$ a contradiction. Since N is graded weakly classical primary and $0 \neq up_1p_2m \in N$, we conclude that either $p_1um \in N$ or $p_2^l(um) \in N$ for some $l \in \mathbb{Z}^+$, and hence either $\frac{p_1}{r}\frac{m}{t} = \frac{up_1m}{urt} \in S^{-1}N$ or $(\frac{p_2}{s})^l\frac{m}{t} = \frac{up_2^l m}{us^l t} \in S^{-1}N$. Thus $S^{-1}N$ is a graded weakly classical primary submodule of $S^{-1}M$.

(ii) Suppose that $S^{-1}N$ is a graded weakly classical primary submodule of $S^{-1}M$ such that $G\text{-}Zdv_R(N) \bigcap S = \phi$ and $G\text{-}Zdv_R(M/N) \bigcap S = \phi$. Let $p_1, p_2 \in h(R)$ and $m \in h(M)$ such that $0 \neq p_1p_2m \in N$. Then $\frac{p_1}{1}\frac{p_2}{1}\frac{m}{1} \in S^{-1}N$. If $\frac{p_1}{1}\frac{p_2}{1}\frac{m}{1} = 0$, then there exists $v \in S$ such that $vp_1p_2m = 0$ that contradicts $G\text{-}Zdv_R(N) \bigcap S = \phi$. Since $S^{-1}N$ is a graded weakly classical primary submodule of $S^{-1}M$ and $0 \neq \frac{p_1}{1}\frac{p_2}{1}\frac{m}{1} \in S^{-1}N$, we conclude that either $\frac{p_1}{1}\frac{m}{1} \in S^{-1}N$ or $(\frac{p_2}{1})^t\frac{m}{1} \in S^{-1}N$ for some $t \in \mathbb{Z}^+$. If $\frac{p_1}{1}\frac{m}{1} \in S^{-1}N$, then there exists $s \in S$ such that $sp_1m \in N$ and since $G\text{-}Zdv_R(M/N) \bigcap S = \phi$, $p_1m \in N$. If $(\frac{p_2}{1})^t\frac{m}{1} \in S^{-1}N$, then there exists $w \in S$ such that $wp_2^t m \in N$ and since $G\text{-}Zdv_R(M/N) \bigcap S = \phi$, we have $p_2^t m \in N$. Therefore, N is a graded weakly classical primary R-submodule of M.

□

Lemma 2.12. *Let R be a G-graded ring, M a graded R-module and N a graded weakly classical primary submodule of M and $g \in G$. Suppose that $r, s \in R_e$ and $m \in M_g$ such that $rsm \in N_g$, $rm \notin N_g$ and $s \notin eGr((N_g :_{R_e} m))$. Then the following hold:*

(i) $rsm = 0$.

(ii) $rsN_g = \{0\}$.

(iii) $r(N_g :_{R_e} M_g)m = s(N_g :_{R_e} M_g)m = \{0\}$.

(iv) $r(N_g :_{R_e} M_g)N_g = s(N_g :_{R_e} M_g)N_g = \{0\}$.
(v) $(N_g :_{R_e} M_g)^2 m = \{0\}$.

Proof. By Theorem 2.6, N_g is a weakly classical g-primary R_e-submodule of M_g for every $g \in M_g$.

(i) Since $rsm \in N_g$, $rm \notin N_g$ and $s \notin eGr((N_g :_{R_e} m))$, we conclude that $rsm = 0$

(ii) Assume that $rsN_g \neq 0$, then there is an element $n \in N_g$ such that $rsn \neq 0$, so $0 \neq rs(m+n) = rsn \in N_g$, we conclude that $r(m+n) \in N_g$ or $s \in eGr((N_g :_{R_e} m+n))$. Thus $rm \in N_g$ or $s \in eGr((N_g :_{R_e} m))$ which is impossible, consequently $rsN_g = \{0\}$.

(iii) Assume that $r(N_g :_{R_e} M_g)m \neq \{0\}$, then there is an element $w \in (N_g :_{R_e} M_g)$ such that $rwm \neq 0$.By part (i), we have $0 \neq rwm = r(s+w)m \in N_g$. Since N_g is a weakly classical g-primary R_e-submodule of M_g, we have either $rm \in N_g$ or $(s+w) \in eGr((N_g :_{R_e} m))$. Hence either $rm \in N_g$ or $s \in eGr((N_g :_{R_e} m))$, which is impossible, consequently $r(N_g :_{R_e} M_g)m = \{0\}$. With a same argument, we can show that $s(N_g :_{R_e} M_g)m = \{0\}$

(iv) Assume that $r(N_g :_{R_e} M_g)N_g \neq \{0\}$, then $rka \neq 0$ for some $k \in (N_g :_{R_e} M_g)$ and $a \in N_g$. By parts (ii) and (iii), we conclude that $0 \neq r(s+k)(m+a) = rka \in N_g$. Since N_g is a weakly classical g-primary R_e-submodule of M_g, we have either $r(m+a) \in N_g$ or $(s+k) \in eGr((N_g :_{R_e} m+a)$. Hence either $rm \in N_g$ or $s \in eGr((N_g :_{R_e} m))$ which is impossible, consequently $r(N_g :_{R_e} M_g)N_g = \{0\}$. With a same argument, we can show that $s(N_g :_{R_e} M_g)N_g = \{0\}$.

(v) Assume that $(N_g :_{R_e} M_g)^2 m \neq \{0\}$. Then $m_1 m_2 m \neq 0$ for some $m_1, m_2 \in (N_g :_{R_e} M_g)$. By part(iii), we have $0 \neq m_1 m_2 m = (r+m_1)(s+m_2)m \in N_g$. Since N_g is a weakly classical g-primary R_e-submodule of M_g, we have either $(r+m_1)m \in N_g$ or $(s+m_2) \in eGr((N_g :_{R_e} m)$ and hence $rm \in N_g$ or $s \in eGr((N_g :_{R_e} m)$ which is impossible, consequently $(N_g :_{R_e} M_g)^2 m = \{0\}$.

□

Theorem 2.13. *Let R be a G-graded ring, M a graded R-module and N a graded weakly classical primary submodule of M. Then for each $g \in M_g$, either N_g is a classical g-primary R_e-submodule of M_g or $(N_g :_{R_e} M_g)^2 N_g = 0$.*

Proof. By Theorem 2.6 , N_g is a weakly classical g-primary R_e-submodule of M_g for every $g \in M_g$. It is enough to show that if $(N_g :_{R_e} M_g)^2 N_g \neq 0$ for some $g \in G$, then N_g is a classical g-primary R_e-submodule of M_g. Assume that $(N_g :_{R_e} M_g)^2 N_g \neq 0$ for some $g \in G$. Then there are $w_1, w_2 \in (N_g :_{R_e} M_g)$ and $n \in N_g$ such that $w_1 w_2 n \neq 0$. Now assume that N_g is not classical g-primary R_e-submodule of M_g. Then there exist $r, s \in R_e$ and $m \in M_g$ such that $rsm \in N_g$, $rm \notin N_g$ and $s \notin eGr((N_g :_{R_e} m))$. By Lemma 2.12, we have $0 \neq (r+w_1)(s+w_2)(m+n) = w_1 w_2 n \in N_g$. Since N_g is a weakly classical g-primary R_e-submodule of M_g, either $(r+w_1)(m+n) \in N_g$ or $(s+w_2) \in eGr((N_g :_{R_e} m+n)$. Hence either $rm \in N_g$ or $s \in eGr((N_g :_{R_e} m)$, a contradiction. □

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